Advances and Applications in Mechanical Engineering and Technology
Volume 2, Number 1/2, 2010, Pages 59-71

SOLUTION TO TIME FRACTIONAL HEAT-EQUATION USING ONE-DIMENSIONAL LAPLACE TRANSFORMS

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Abstract

In this work, authors present a new theorem and corollary on multi-dimensional Laplace transformations. They also develop some applications based on this results. The one-dimensional Laplace transformation is useful to obtain the solution of partial fractional differential equations.

1. Introduction and Notation

Engineering and other areas of sciences can be successfully modelled by the use of fractional derivatives. That is because of the fact that, a realistic modelling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. Fractional differential equations arise in unification of diffusion and wave propagation phenomenon. The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative by a fractional derivative of order α, $0 < \alpha \leq 1$.

Keywords and phrases: one-dimensional Laplace transform, partial fractional differential equations, heat equations, wave equation.

Received July 19, 2010

In the next part of this section, we consider the time fractional heat equation (time fractional in the-Caputo sense).

In this section, we consider methods and results for the partial fractional diffusion equation, which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1], [2], [3], [11], the Fourier transform method [10], the iteration method [17], and the operational method [10]. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. More detailed information about some of these results can be found in a survey paper by Kilbas and Trujillo [9]. Atanackovic and Stankovic [4, 5] used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model far a viscoelastic rod. Oldham and Spanier [12] and [13], respectively, by reducing a boundary value problem involving Fick's second low in electroanalytic chemistry to a formulation based on the partial Riemann-Liouville fractional with half derivative. Oldham and Spanier [13] gave other application of such equations for diffusion problems. Wyss [20] and Schneider [18] considered the time fractional diffusion and wave equations, and obtained the solution in terms of Fox functions.

2. Solution to Non-Homogeneous Partial Fractional Differential Equation (Heat Equation)

$$ {}^{c}D_t^{\alpha} u = a\frac{\partial^2 u}{\partial x^2} + f(t) + \int_0^x g(\lambda)d\lambda - ku, \tag{1.1}$$

where $t > 0$, $0 < x < l$, a, λ are constants. The boundary conditions are

$$u(x, 0) = 0, \quad u(0, t) = 0, \quad u(l, t) = 0.$$

Solution.

$$ {}^{c}D_t^{\alpha} u - a\frac{\partial^2 u}{\partial x^2} + ku = f(t) + \int_0^x g(\lambda)d\lambda, \quad 0 < \alpha \le 1.$$

Taking one dimensional Laplace transform of both sides of the Equation (1.1) with respect to t (with assumption that $\alpha = \frac{1}{2}$) [15]. We have:

$$\overline{U}''(x, s) - \left(\frac{\sqrt{s} + k}{a}\right)\overline{U}(x, s) = -\frac{1}{a}\left\{F(s) + \frac{1}{s}\int_0^x g(\lambda)d\lambda\right\}. \tag{1.2}$$

We obtain

$$\overline{U}(x, s) = c_1 e^{\sqrt{\frac{\sqrt{s}+k}{a}}x} + c_2 e^{-\sqrt{\frac{\sqrt{s}+k}{a}}x} + \frac{1}{\sqrt{s} + k}\left\{F(s) + \frac{1}{s}\int_0^x g(\lambda)d\lambda\right\}. \tag{1.3}$$

By using the boundary conditions, one gets the unknown constants c_1, c_2 as follows:

$$c_1 = \frac{F(s)}{\sqrt{s} + k}\,\frac{1 - e^{-\sqrt{\frac{\sqrt{s}+k}{a}}l}}{e^{-\sqrt{\frac{\sqrt{s}+k}{a}}l} - e^{\sqrt{\frac{\sqrt{s}+k}{a}}l}} + \frac{1}{s(\sqrt{s} + k)}\,\frac{\int_0^l g(\lambda)d\lambda}{e^{-\sqrt{\frac{\sqrt{s}+k}{a}}l} - e^{\sqrt{\frac{\sqrt{s}+k}{a}}l}},$$

$$c_2 = \frac{F(s)}{\sqrt{s} + k}\,\frac{e^{\sqrt{\frac{\sqrt{s}+k}{a}}l} - 1}{e^{-\sqrt{\frac{\sqrt{s}+k}{a}}l} - e^{\sqrt{\frac{\sqrt{s}+k}{a}}l}} - \frac{1}{s(\sqrt{s} + k)}\,\frac{\int_0^l g(\lambda)d\lambda}{e^{-\sqrt{\frac{\sqrt{s}+k}{a}}l} - e^{\sqrt{\frac{\sqrt{s}+k}{a}}l}}.$$

Therefore, relation (1.3) has the following form:

$$\overline{U}(x, s) = \frac{F(s)}{\sqrt{s} + k}\left\{\frac{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}(x - l)}{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}l} - \frac{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}x}{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}l} + 1\right\}$$

$$-\left(\int_0^l g(\lambda)d\lambda\right)\frac{1}{\sqrt{s} + k}\left(\frac{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}x}{s\sinh\sqrt{\frac{\sqrt{s} + k}{a}}l}\right). \tag{1.4}$$

We may rewrite (1.4) as follows:

$$\overline{U}(x, s) = \frac{sF(s) - f(0)}{\sqrt{s} + k}\left\{\frac{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}(x - l)}{s\sinh\sqrt{\frac{\sqrt{s} + k}{a}}l} - \frac{\sinh\sqrt{\frac{\sqrt{s} + k}{a}}x}{s\sinh\sqrt{\frac{\sqrt{s} + k}{a}}l} + \frac{1}{s}\right\}$$

$$-(f(0)+\int_0^l g(\lambda)d\lambda)\frac{1}{\sqrt{s}+k}\left(\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right)$$

$$+\frac{f(0)}{\sqrt{s}+k}\left\{\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(x-l)}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}+\frac{1}{s}\right\}. \tag{1.5}$$

At this point, we take inverse Laplace transform of (1.5) term wise. At first, evaluating the inverse Laplace transform of the term $\frac{1}{\sqrt{s}+k}$,

$$L^{-1}\left\{\frac{1}{\sqrt{s}+k}\right\}=L^{-1}\left\{\frac{1}{\sqrt{s}}\frac{1}{1+\frac{k}{\sqrt{s}}}\right\}=L^{-1}\left\{\frac{1}{\sqrt{s}}\left(1-\frac{k}{\sqrt{s}}+\frac{k^2}{s}-\cdots\right)\right\}$$

$$=L^{-1}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n k^n}{s^{\frac{n+1}{2}}}\right\}=\sum_{n=0}^{\infty}\frac{(-1)^n k^n}{\Gamma\left(\frac{n+1}{2}\right)}t^{\frac{n-1}{2}}. \tag{1.6}$$

By using relation (1.6), we obtain:

$$L^{-1}\left\{\frac{sf(s)-f(0)}{\sqrt{s}+k}\right\}=L^{-1}\{sF(s)-f(0)\}*L^{-1}\left\{\frac{1}{\sqrt{s}+k}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n k^n}{\Gamma\left(\frac{n+1}{2}\right)}\int_0^t f'(t-\eta)\eta^{\frac{n-1}{2}}d\eta. \tag{1.7}$$

If the first term on the left side of (1.5) is called $H_1(s)$, then one has,

$$H_1(s)=\frac{1}{s}-\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}-\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}$$

$$=\frac{1}{\sqrt{s}}\left\{\frac{1}{\sqrt{s}}-\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}-\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right\}, \tag{1.8}$$

and we also set $P_1(\sqrt{s})$ in the following form:

$$P_1(\sqrt{s}) = \frac{1}{\sqrt{s}} - \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l} - \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}.$$

By replacing $\sqrt{s}$ by s, we have:

$$P_1(s) = \frac{1}{s} - \frac{\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l} - \frac{\sinh\sqrt{\frac{s+k}{a}}(l-x)}{s\sinh\sqrt{\frac{s+k}{a}}l}. \tag{1.9}$$

Now, upon using residue theorem, we may find the inverse Laplace transform of the term $P_1(s)$. First, we find the $L^{-1}\left\{\frac{\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l}\right\}$. We have

$$s\sinh\sqrt{\frac{s+k}{a}}l = 0 \Rightarrow \begin{cases} s = 0, \\ s = s_m = -\left(\frac{m^2\pi^2 a}{l^2} + k\right). \end{cases}$$

Let us calculate residue at $s = 0$, that is,

$$\lim_{s\to 0}(s-0)\frac{e^{st}\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l} = \frac{\sinh\sqrt{\frac{k}{a}}x}{\sinh\sqrt{\frac{k}{a}}l},$$

and residues at $s = s_m$, $m = 0, 1, 2, 3, \cdots$ are

$$\lim_{s\to s_m}(s-s_m)\frac{e^{st}\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l} = \lim_{s\to s_m}\left\{\frac{e^{st}\sinh\sqrt{\frac{s+k}{a}}x}{s}\right\} \times \lim_{s\to s_m}\left\{\frac{1}{\sinh\sqrt{\frac{s+k}{a}}l}\right\}$$

$$= \frac{e^{-\left(k+\frac{m^2\pi^2 a}{l^2}\right)t}\sinh\left(\frac{m\pi i}{l}\right)x}{-\left(k+\frac{m^2\pi^2 a}{l^2}\right)} \times \frac{1}{\frac{l}{2\sqrt{a}}\frac{l}{m\pi i\sqrt{a}}\cosh(m\pi i)}$$

$$= \frac{2m\pi a(-1)^m}{(kl^2 + m^2\pi^2 a)} e^{-\left(k + \frac{m^2\pi^2 a}{l^2}\right)t} \sin\left(\frac{m\pi}{l} x\right),$$

so that

$$L^{-1}\left\{\frac{\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l}\right\}$$

$$= \frac{\sinh\sqrt{\frac{k}{a}}x}{\sinh\sqrt{\frac{k}{a}}l} + 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{(kl^2 + m^2\pi^2 a)} e^{-\frac{m^2\pi^2 a}{l^2}t} \sin\left(\frac{m\pi}{l} x\right). \tag{1.10}$$

If we replace x by $(l - x)$ in (1.10), then:

$$L^{-1}\left\{\frac{\sinh\sqrt{\frac{s+k}{a}}(l-x)}{s\sinh\sqrt{\frac{s+k}{a}}l}\right\} = \frac{\sinh\sqrt{\frac{k}{a}}(l-x)}{\sinh\sqrt{\frac{k}{a}}l}$$

$$+ 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{(kl^2 + m^2\pi^2 a)} e^{-\frac{m^2\pi^2 a}{l^2}t} \sin\left(\frac{m\pi}{l}(l - x)\right). \tag{1.11}$$

Therefore,

$$p_1(t)$$

$$= L^{-1}\{P_1(s)\}$$

$$= 1 - \left\{\frac{\sinh\sqrt{\frac{k}{a}}x}{\sinh\sqrt{\frac{k}{a}}l} + 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{(kl^2 + m^2\pi^2 a)} e^{-\frac{m^2\pi^2 a}{l^2}t} \sin\left(\frac{m\pi}{l} x\right)\right\}$$

$$-\left\{\frac{\sinh\sqrt{\frac{k}{a}}(l-x)}{\sinh\sqrt{\frac{k}{a}}l} + 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{(kl^2 + m^2\pi^2 a)} e^{-\frac{m^2\pi^2 a}{l^2}t}(-1)^{m+1}\sin\left(\frac{m\pi}{l}x\right)\right\}$$

$$= 1 - \frac{\sinh\sqrt{\frac{k}{a}}x}{\sinh\sqrt{\frac{k}{a}}l} - \frac{\sinh\sqrt{\frac{k}{a}}(l-x)}{\sinh\sqrt{\frac{k}{a}}l}$$

$$- 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)} e^{-\frac{m^2\pi^2 a}{l^2}t}\sin\left(\frac{m\pi}{l}x\right). \tag{1.12}$$

Using the following formula, we arrive at the inverse Laplace transform of $H_1(s)$.

$$h_1(t) = L^{-1}\left\{\frac{1}{\sqrt{s}}P_1\left(\sqrt{s}\right)\right\} = \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{\nu^2}{4t}\right)p_1(\nu)d\nu$$

$$= (1 - \frac{\sinh\sqrt{\frac{k}{a}}t}{\sinh\sqrt{\frac{k}{a}}l} - \frac{\sinh\sqrt{\frac{k}{a}}(l-t)}{\sinh\sqrt{\frac{k}{a}}l}) \times \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{x^2}{4t}\right)dx$$

$$- \frac{4\pi a}{\sqrt{\pi t}}\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)}$$

$$\times \int_0^{\infty}\exp\left(-\frac{x^2}{4t}\right)e^{-kx-\frac{m^2\pi^2 a}{l^2}x}\sin\left(\frac{m\pi}{l}t\right)dx$$

$$= (1 - \frac{\sinh\sqrt{\frac{k}{a}}t}{\sinh\sqrt{\frac{k}{a}}l} - \frac{\sinh\sqrt{\frac{k}{a}}(l-t)}{\sinh\sqrt{\frac{k}{a}}l}) \times \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{x^2}{4t}\right)dx$$

$$- 4a\sqrt{\frac{\pi}{t}}\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)}$$

$$\times \sin\left(\frac{m\pi}{l}t\right)\int_0^{\infty}e^{-(\frac{x}{2\sqrt{t}}+(k+\frac{m^2\pi^2 a}{l^2})\sqrt{t})^2+(k+\frac{m^2\pi^2 a}{l^2})^2 t}dx,$$

in the first integral, if we set $\frac{x}{2\sqrt{t}} = u$ and in the second integral $\frac{x}{2\sqrt{t}} + \left(k + \frac{m^2\pi^2 a}{l^2}\right)\sqrt{t} = w,$ then one has:

$$h_1(t) = (1 - \frac{\sinh\sqrt{\frac{k}{a}}t}{\sinh\sqrt{\frac{k}{a}}l} - \frac{\sinh\sqrt{\frac{k}{a}}(l-t)}{\sinh\sqrt{\frac{k}{a}}l})$$

$$-4\pi a\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)}e^{(k+\frac{m^2\pi^2 a}{l^2})^2 t}\sin\left(\frac{m\pi}{l}t\right)erfc\left((k+\frac{m^2\pi^2 a}{l^2})\sqrt{t}\right). \tag{1.13}$$

Similarly, we get $h_2(t)$ as follows:

$$L^{-1}\{H_2(s)\} = L^{-1}\left\{\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right\} = \left\{\frac{1}{\sqrt{s}}\frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right\}.$$

If $P_2\left(\sqrt{s}\right) = \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}x}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}$, then $P_2(s) = \frac{\sinh\sqrt{\frac{s+k}{a}}x}{s\sinh\sqrt{\frac{s+k}{a}}l}$, where its inverse Laplace transform is as follows:

$$p_2(t) = \frac{\sinh\sqrt{\frac{k}{a}}x}{\sinh\sqrt{\frac{k}{a}}l} + 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{\left(kl^2+m^2\pi^2 a\right)}e^{-\frac{m^2\pi^2 a}{l^2}t}\sin\left(\frac{m\pi}{l}x\right). \tag{1.14}$$

Therefore,

$$h_2(t) = L^{-1}\left\{\frac{1}{\sqrt{s}}P_2\left(\sqrt{s}\right)\right\} = \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{\nu^2}{4t}\right)p_2(\nu)d\nu$$

$$= \frac{\sinh\sqrt{\frac{k}{a}}t}{\sinh\sqrt{\frac{k}{a}}l}\int_0^{\infty}\exp\left(-\frac{x^2}{4t}\right)dx$$

$$+\frac{4\pi a}{\sqrt{\pi t}}\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)}\sin\left(\frac{m\pi}{l}t\right)\int_0^{\infty}\exp\left(-\frac{x^2}{4t}\right)e^{-kx-\frac{m^2\pi^2 a}{l^2}x}dx.$$

Letting $\frac{x}{2\sqrt{t}} = u$ in the first integral and $\frac{x}{2\sqrt{t}} + \left(k + \frac{m^2\pi^2 a}{l^2}\right)\sqrt{t} = w$ in the second integral, we get:

$$h_2(t) = \frac{\sinh\sqrt{\frac{k}{a}}t}{\sinh\sqrt{\frac{k}{a}}l}$$

$$+ 4\pi a\sum_{m=0}^{\infty}\frac{m((-1)^m+1)}{\left(kl^2+m^2\pi^2 a\right)}e^{(k+\frac{m^2\pi^2 a}{l^2})^2 t}\sin\left(\frac{m\pi}{l}t\right)erfc\left((k+\frac{m^2\pi^2 a}{l^2})\sqrt{t}\right), \tag{1.15}$$

we obtain $h_3(t)$ by similar method of calculation,

$$H_3(s) = \left\{\frac{1}{s} - \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{s\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right\} = \frac{1}{\sqrt{s}}\left\{\frac{1}{\sqrt{s}} - \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}\right\}. \tag{1.16}$$

Setting $P_3\left(\sqrt{s}\right) = \frac{1}{\sqrt{s}} - \frac{\sinh\sqrt{\frac{\sqrt{s}+k}{a}}(l-x)}{\sqrt{s}\sinh\sqrt{\frac{\sqrt{s}+k}{a}}l}$ and replacing $\sqrt{s}$ by s, then one gets

$$P_3(s) = \frac{1}{s} - \frac{\sinh\sqrt{\frac{s+k}{a}}(l-x)}{s\sinh\sqrt{\frac{s+k}{a}}l}.$$

$$p_3(t) = 1 - \frac{\sinh\sqrt{\frac{k}{a}}(l-x)}{\sinh\sqrt{\frac{k}{a}}l}$$

$$- 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{\left(kl^2 + m^2\pi^2 a\right)} e^{-\frac{m^2\pi^2 a}{l^2}t} \sin\left(\frac{m\pi}{l}(l-x)\right)$$

$$= 1 - \frac{\sinh\sqrt{\frac{k}{a}}(l-x)}{\sinh\sqrt{\frac{k}{a}}l}$$

$$+ 4\pi a e^{-kt}\sum_{m=0}^{\infty}\frac{m(-1)^m}{\left(kl^2 + m^2\pi^2 a\right)} e^{-\frac{m^2\pi^2 a}{l^2}t} \sin\left(\frac{m\pi}{l}x\right). \tag{1.17}$$

$$h_3(t) = L^{-1}\left\{\frac{1}{\sqrt{s}}P_3\left(\sqrt{s}\right)\right\} = \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{\nu^2}{4t}\right)p_3(\nu)d\nu$$

$$= (1 - \frac{\sinh\sqrt{\frac{k}{a}}(l-t)}{\sinh\sqrt{\frac{k}{a}}l}) \times \frac{1}{\sqrt{\pi t}}\int_0^{\infty}\exp\left(-\frac{\nu^2}{4t}\right)d\nu$$

$$+ \frac{4\pi a}{\sqrt{\pi t}}\sum_{m=0}^{\infty}\frac{m(-1)^m}{\left(kl^2 + m^2\pi^2 a\right)}$$

$$\times \sin\left(\frac{m\pi}{l}x\right)\int_0^{\infty}\exp\left(-\frac{\nu^2}{4t}\right)e^{-k\nu - \frac{m^2\pi^2 a}{l^2}\nu}d\nu.$$

As before, supposing $\frac{x}{2\sqrt{t}} = u$ in the first integral and $\frac{x}{2\sqrt{t}} + \left(k + \frac{m^2\pi^2 a}{l^2}\right)\sqrt{t} = w$ in the second integral, then:

$$h_3(t) = (1 - \frac{\sinh\sqrt{\frac{k}{a}}(l-t)}{\sinh\sqrt{\frac{k}{a}}l})$$

$$+ 4\pi a \sum_{m=0}^{\infty} \frac{m(-1)^m}{\left(kl^2 + m^2\pi^2 a\right)} e^{(k+\frac{m^2\pi^2 a}{l^2})^2 t}$$

$$\times \sin\left(\frac{m\pi}{l} t\right) erfc\left((k + \frac{m^2\pi^2 a}{l^2})\sqrt{t}\right). \tag{1.18}$$

Finally, the inverse Laplace transform of (1.5) is:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n k^n}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^t (1 - \frac{\sinh\sqrt{\frac{k}{a}}(t-\xi)}{\sinh\sqrt{\frac{k}{a}}l}$$

$$- \frac{\sinh\sqrt{\frac{k}{a}}(l-t-\xi)}{\sinh\sqrt{\frac{k}{a}}l})(\int_0^{\xi} f'(\xi-\eta)\eta^{\frac{n-1}{2}} d\eta) d\xi$$

$$- 4\pi a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m((-1)^{m+n} + (-1)^n)k^n}{\Gamma\left(\frac{n+1}{2}\right)\left(kl^2 + m^2\pi^2 a\right)} \int_0^t e^{(k+\frac{m^2\pi^2 a}{l^2})^2 (t-\xi)}$$

$$\times \sin\left(\frac{m\pi}{l}(t-\xi)\right) erfc\left((k + \frac{m^2\pi^2 a}{l^2})\sqrt{t-\xi}\right)(\int_0^{\xi} f'(\xi-\eta)\eta^{\frac{n-1}{2}} d\eta) d\xi$$

$$- (f(0) + \int_0^l g(\lambda) d\lambda)\{\sum_{n=0}^{\infty} \frac{(-1)^n k^n}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^t (t-\xi)^{\frac{n-1}{2}} \frac{\sinh\sqrt{\frac{k}{a}}\xi}{\sinh\sqrt{\frac{k}{a}}l} d\xi$$

$$+ 4\pi a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m((-1)^{m+n} + (-1)^n)k^n}{\Gamma\left(\frac{n+1}{2}\right)\left(kl^2 + m^2\pi^2 a\right)} \int_0^t (t-\xi)^{\frac{n-1}{2}}$$

$$\times e^{(k+\frac{m^2\pi^2 a}{l^2})^2 \xi} \sin\left(\frac{m\pi}{l}\xi\right) erfc\left((k + \frac{m^2\pi^2 a}{l^2})\sqrt{\xi}\right) d\xi\}$$

$$+ f(0)\{\sum_{n=0}^{\infty}\frac{(-1)^n k^n}{\Gamma\left(\frac{n+1}{2}\right)}\int_0^t (t-\xi)^{\frac{n-1}{2}}(1-\frac{\sinh\sqrt{\frac{k}{a}}(l-\xi)}{\sinh\sqrt{\frac{k}{a}}l})d\xi$$

$$+ 4\pi a\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m(-1)^{m+n}k^n}{\Gamma\left(\frac{n+1}{2}\right)\left(kl^2+m^2\pi^2 a\right)}$$

$$\times\int_0^t (t-\xi)^{\frac{n-1}{2}}\times e^{(k+\frac{m^2\pi^2 a}{l^2})^2\xi}\sin\left(\frac{m\pi}{l}\xi\right)erfc\left((k+\frac{m^2\pi^2 a}{l^2})\sqrt{\xi}\right)d\xi\}.$$

3. Conclusion

The paper is devoted to study and application of one-dimensional Laplace transform. The one-dimensional Laplace transform provides powerful method for analyzing linear systems. The main purpose of this work is to develop a method for finding analytic solution of the time fractional heat equation.

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